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Relativistic structure of white dwarfs

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Abstract An analytic approach has been used to study the special relativistic features of degenerate configurations such as the white dwarfs, governed by the equation of state $P = Af(x)$ (A = constant = $\pi m^4 c^5/3h^3$). Approximate analytical solutions to the equations of hydrostatic equilibrium, suitable for use in short computer programmes or on small calculators, have been given. Results of our calculations for few physical parameters are given in tabular and graphical forms. Our approximate analytical results are very close to the previous numerically calculated values.

Keywords White dwarfs, relativistic structure, analytical solutions

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1. Introduction

Structural properties of a star (or a polytropic star) and particularly of white dwarfs, have been a fascinating subject of study to applied mathematicians and to astrophysicists for long ($\simeq 10^3$ yrs). As pointed out by Chandrasekhar [1], the theory of polytropes is the most important contribution which "stellar structure" has made to Applied Mathematics. The main objects of the studies in this direction are to : (i) derive the complete march of physical variables, such as, the density ρ, the temperature T, *etc.*, (ii) describe quantitatively the kind of steady state prevailing inside the star, and (iii) evaluate the fundamental physical processes for setting up the steady state.

Most of the stars in the sky can be studied in Newtonian physics. Such Newtonian stars are of importance and certainly deserve careful attention from the viewpoint that they become the limiting cases for the most exotic objects of interest to the general relativists. White dwarfs can be distinguished from other stars fundamentally by their "luminosity" and "faintness", such as the companion of Sirius is well recognized by its high values of the density (10^6 gm cm^{-3} and even 10^8 gm cm^{-3}) and smaller size than those of stars on the main series. Because of their higher temperature, they are "whiter" and named "white dwarfs".

The clue to the understanding of the structure of white dwarfs was first discovered by Fowler [2] who pointed out that the electron assembly inside the stars must be degenerate in the sense of Fermi-Dirac statistics. Using suitable equations of state, several authors have discussed the structure of the stellar models or of white dwarfs under the treatments of Newtonian [2–9], special relativistic [10–12] and general relativistic [13–17] theories in a variety of density ranges $10^3 \le \rho \le 10^4$, $10^5 \le \rho \le 10^{11}$ and $\rho > 10^{10}$ (gm cm^{-3}), respectively. As general relativity predicts stronger gravitational forces in astrophysical bodies, it provides a better treatment to the problems in situations (free neutron regime between the density range $\rho \simeq 4.3 \times 10^{11}$ gm cm^{-3} and 2.4×10^{14} gm cm^{-3}, and beyond $\rho \sim 1.5 \times 10^{15}$ gm cm^{-3}, where the physics of matter is poorly understood) where the Newtonian theory fails.

The theory of pressure ionization has been put forward by many eminent workers [18–20] in the study of the internal constitution of non-relativistic polytropic stars, massive stars, white dwarfs and planetary bodies, as well as in the general relativistic problems [13,21–24] for describing equations of state of cold, catalyzed matter in different density ranges from $\rho \simeq 10^4$ gm cm^{-3} to the highest densities. White dwarfs and planets are well-approximated by polytropes with $\sim 4/3 \le \rho \le 5/3$. Ramsey [25], Bullen [26] Brown [27], Mestel [28] and Inglis [29] have made a detailed study of the internal constitution of planets and white dwarfs, which later on, were followed by several other contributions. For example in our previous work [30,31], we considered the effect of nuclear size correction to the relativistic Thomas Fermi Model and the limiting mass of dense stellar matter. The theory of rotating white dwarfs can be viewed as well advanced. The early works of James [32] (on rotating degenerate polytropes),

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Ostriker [33,34] and Durisen [35] (on differentially rotating degenerate configurations in Newtonian theory) contain very nice treatments of the structures and stability properties of white dwarfs.

In the above works, and particularly, in special relativistic problems [1,11,12] with which we are presently concerned, the methods generally adopted to solve the equilibrium equations are : (i) perturbation approach, (ii) variational principle, (iii) formation of self-consistent density and potential distributions, and (iv) numerical methods. In the applications of these methods, one is generally faced with the difficulties that these are, however, lengthy, cumbersome and involve mathematical complexities from computational view-point. Clearly previous numerical method are not also economical for the computer programming. To avoid all such difficulties, an easier and shorter method called Pade' (2,2) approximation technique has been employed, as used elsewhere [17,35-37] for analytically solving the equilibrium equations. We find that one may compute the gravitational potential ϕ (eq 8) for an arbitrary value of the dimensionless radial distance η with utmost ease and the boundary value η_1 can be obtained from the biquadratic equation (10) without the need of computations of long tables.

In Section 2, special relativistic structure equations are briefly described. Section 3 deals with the approximate analytical solutions leading to the description of the physical structure of the white dwarfs. Results of our calculations are presented in tabular [Tables 1 and 2] and graphical forms

Table 1. Comparison of approximate analytical values with sophisticated numerical values

y_0^{-2}	η_1	$\eta_1^2\phi'(\eta)$	$\rho_0/\bar{\rho}$
0 00	6 8968*	2 0182*	54 1820*
	6 9211	1 9918	54 4820
	$3\ 523 \cdot 10^{-3}$**	0 0130**	$5\ 537 \cdot 10^{-3}$
0 01	5 3571*	1 9321*	26 203*
	5 5881	1 9230	27 015
	0 043**	0 005**	00 031**
0 02	4 9857*	1 8652*	21 486*
	5 2880	2 3441	22 603
	0 0606**	0 2568**	00 052**
0 05	4 4601*	1 7096*	16 018**
	4 7379	1 8126	16 883
	0 0623**	0 060**	00 054**
0 10	4 0690*	1 5186*	12 626*
	4 3599	2 2528	13 346
	0 0715**	0 4835**	00 057**
0 20	3 7271*	1 2430*	09 9348*
	4 0372	1 8650	10 571
	0 0832**	0 5012**	00 064**

Table 2. Comparison of approximate analytical values of M/M_3 with its sophisticated numerical values

y_0^{-2}	M/M_3	R/l_1
0 00	1 0000*	0 0000*
	1 0133	0 0000
	0 0133*	----
0 01	0 9573**	0.5357*
	0 9528	0 5588
	0 0047**	0 0430**
0 02	0 92419*	0 70508*
	1 1615	0 7478
	0 2567**	0 0606**
0 05	0 84709*	0 9973*
	0 89810	1 0594
	0 0602**	0 0623**
0 10	0 7524*	0 2867*
	1 1162	1 3787
	0 4835**	0 0715**
0 20	0 6185*	1 6668*
	0 9241	1 8055
	0 5006**	0 0832**

*Chandrasekhar's [1] values
**Relative Error R E

[Figures (1-3)] which are found to be in good agreement with those of numerical methods. Section 4 describes some important structural parameters. Discussion on the velocity of sound is given in section 5.

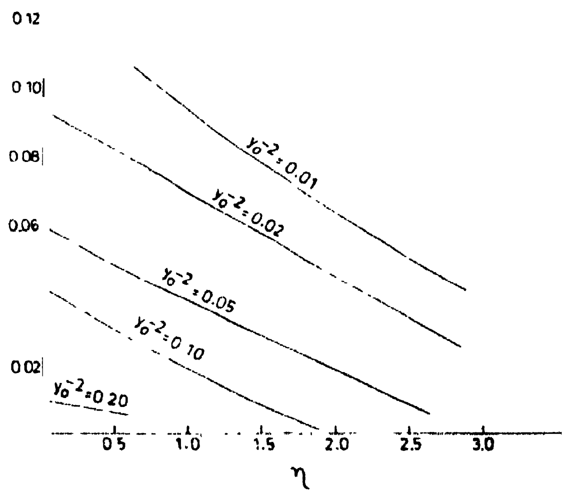

Figure 1. The $(v/c, \eta)$ curves plotted for small electronic concentrations $x_0 \to 0$ Each curve is labelled by the corresponding value of y_0^{-2}

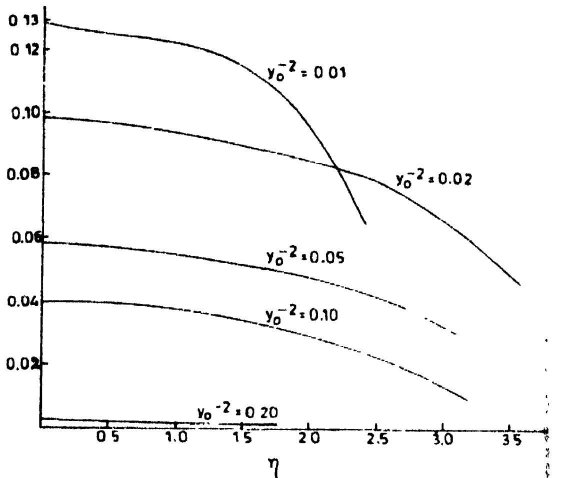

Figure 2. The $(v/c, \eta)$ curves plotted for large electronic concentrations $x_0 \to \infty$. Each curve is labelled for the corresponding value of y_0^{-2}

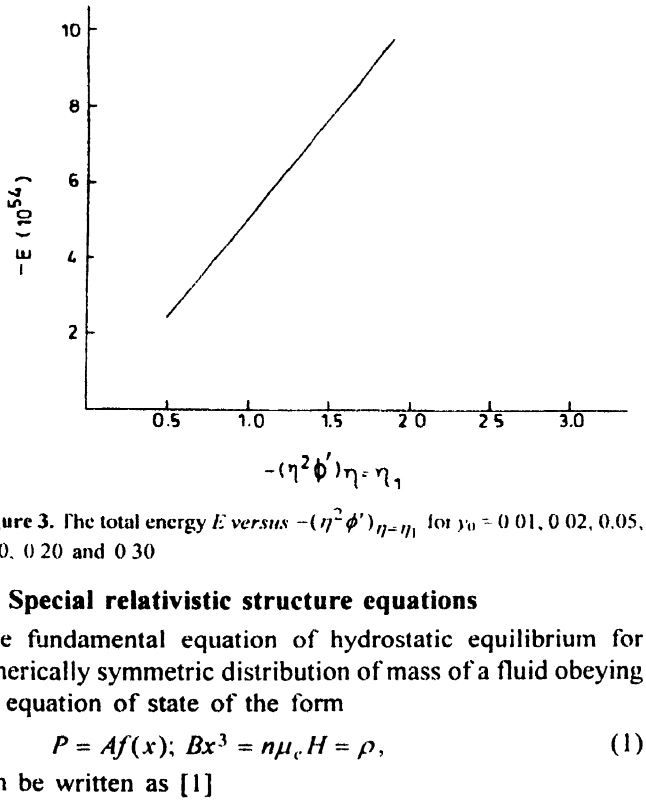

Figure 3. The total energy E *versus* $-(\eta^2\phi')_{\eta=\eta_1}$ for $y_0 = 0.01, 0.02, 0.05, 0.10, 0.20$ and 0.30

2. Special relativistic structure equations

The fundamental equation of hydrostatic equilibrium for spherically symmetric distribution of mass of a fluid obeying an equation of state of the form

$$P = Af(x);\ Bx^3 = n\mu_e H = \rho, \tag{1}$$

can be written as [1]

$$\frac{A}{B}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi G B x^3 \tag{2}$$

where $f(x) = x(2x^2-3)(x^2+1)^{\frac12} + 3\sinh^{-1}x$;

n = electronic concentration $= 8\pi m^3c^3x^3/3h^3$.

P, ρ and r denote the total pressure, density and radial distance, respectively ($A = \pi m^4c^5/3h^3 = 6.01 \times 10^{22}$, $B = 8\pi m^3c^3\mu_e H/3h^3$ gm cm^{-3}, h = Planck's constant $= 6.62 \times 10^{-27}$ erg. sec. H = mass of the proton $= 1.672 \times 10^{-24}$ gm. μ_e = mean molecular weight $\simeq 1$, $G = 6.67 \times 10^{-8}$ Dyn-cm^2/gm^2). After the substitutions,

$$y_0^2 = x_0^2 + 1,\ r = \alpha\eta;\ y = y_0\phi;\ \alpha = \left(\frac{2A}{\pi G}\right)^{\frac12}\frac{1}{By_0}, \tag{3}$$

eq. (2) is transformed into the form

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{y_0^2}\right) \tag{4}$$

which satisfies the following boundary conditions :

$$\phi(0) = 1,\ \phi'(\eta) = 0. \tag{5}$$

The boundary η_1 of the configuration is defined by

$$\phi(\eta_1) = y_0^{-1}, \tag{6}$$

where ϕ is the gravitational potential.

3. Approximate analytical solutions of equation (4)

The series solution of eq. (4), containing terms up to η^8, under the boundary conditions (5) can be written as

$$\phi(\eta) = 1 + a_1\eta^2 + a_2\eta^4 + a_3\eta^6 + a_4\eta^8 + \ldots\ldots, \tag{7}$$

where $a_1 = -\frac{q^3}{6}$, $a_2 = \frac{q^4}{40}$, $a_3 = \frac{q^3(5q^2+14)}{7!}$

$$a_4 = -\frac{q^6(339q^2+280)}{3\times 9!};\ q^2 = (y_0^2-1)/y_0^2.$$

Then, it immediately follows that approximate analytical solution (Pade' (2,2) approximation) of eq. (4) can be put in the form of rational function

$$\phi(\eta) = \phi_{22}(\eta) = \frac{1 + A\eta^2 + B\eta^4}{1 + C\eta^2 + D\eta^4}, \tag{8}$$

where A, B, C and D are constants given by

$$A = -a_1 + C,\ B = a_2 - a_1C + D,\ C = (a_2a_3 - a_1a_4)$$

$$D = (a_3^2 - a_2a_4)/\Delta;\ \Delta = a_2^2 - a_1a_3. \tag{9}$$

The geometrical size (boundary value ξ_1) of the configuration is given by the following biquadratic equation

$$(B - Dy_0^{-1})\eta_1^4 + (A - Cy_0^{-1})\eta_1^2 + (1 - y_0^{-1}) = 0. \tag{10}$$

4. Some important structural parameters

(a) The ratio of the mean density to central density :

The ratio of the mean density $\bar{\rho}$ of matter (interior to $r = R = \alpha\eta$) to the central density ρ_0 is given by [1]

$$\frac{\bar{\rho}}{\rho_0} = \left(1 - \frac{1}{y_0^2}\right)^{-\frac32}\frac{3}{\eta_1}(\phi')_{\eta=\eta_1} \tag{11}$$

(b) The mass :

The mass of the whole configuration is given by

$$M(\eta) = 4\pi\left(\frac{2A}{\pi G}\right)^{\frac32}\frac{1}{B^2}(\eta^2\phi')_{\eta=\eta_1}, \tag{12}$$

where ϕ' denotes the derivative of ϕ with respect to the radial distance η.

(c) The ratio of total mass M to the limiting mass M_3 :

The relation $M/M_3 = \frac{\Omega(y_0)}{O\omega_3}$; $\Omega(y_0) = -(\eta^2\phi)_\eta = \eta_1$; (13)

$$O\omega_3 = -(\xi^2\theta_3)\xi = \xi_1(\theta_3); \xi = \sqrt{2}\eta;\ \phi \to \theta_3$$

defines the ratio M/M_3 (M_3= limiting mass of the configuration corresponding to that of the polytropic index 3).

(d) The ratio R/l_1:

An important quantity which measures the ratio of total radius R to the length $l_1(=7.71\times10^8)/\mu_0$ cm; μ_0= central mean molecular weight) is given by

$$\frac{R}{l_1} = \eta y_0^{-1} \tag{14}$$

(e) The total energy E:

Following the fundamental principle of equivalence of mass and energy, the total energy E of the configuration is

$$E = -4\pi\left(\frac{2A}{\pi G}\right)^{\frac{3}{2}} \frac{1}{B^2}\left(\eta^2 \frac{dd}{d\eta}\right)_{n=n_1} c^2 = Mc^2; \tag{15}$$

where c = velocity of light = 2.9978×10^{10} cm/sec and μ_e=mean molecular weight $\cong 1$, for all practical purposes. Obviously, the total energy E is directly proportional to $(\eta^2\phi')_{\eta=\eta_1}$; Figure 3 shows that E increases monotonically with the mass function $-(\eta^2\phi')_{\eta=\eta_1}$ for $y_0^{-2} = 0.01$, 0.02, 0.05, 0.10 and 0.20.

(f) The gravitational potential :

The gravitational potential V of the configuration of total mass M and radius R ($r = R$) is given by [1].

$$V = -\frac{8A}{B} y_0(\phi - y_0^{-1})\frac{GM}{R}, \quad (R \geq r) \tag{16}$$

which can be re-expressed as

$$\frac{8Ay_0}{B}\left[\frac{1}{\eta_1}\left(\eta^2\frac{d\phi}{d\eta}\right)_{\eta=\eta_1} \quad -\frac{1}{y_0}\right] \tag{17}$$

5. Velocity of sound

We shall discuss the velocity of sound inside the configurations for two cases : (i) small electronic concentrations ($x \to 0$) and (ii) large electronic concentrations ($x \to \infty$).

Case (i) For $x \to 0$, $f(x) = 8x^5/5$:

Then, according to the first law of thermodynamics, the general expression for the velocity of sound v_s in the fluid (under the assumption of adiabatic process is given by

$$\frac{v^2 s}{c^2} = \frac{dp}{d\varepsilon}, \quad \varepsilon = \text{energy density} = \rho c^2. \tag{18}$$

Using the equation of state (1), we obtain from the foregoing equation

$$v/c = 0.013468725x^{\frac{1}{2}} = 0.013468725y_0^{\frac{1}{2}}\left(\phi^2 - \frac{1}{y_0^2}\right)^{\frac{1}{2}}. \tag{19}$$

Case (ii) For $x \to \infty$, $f(x) = 2x^4$:

Hence, we obtain from (1) and (18);

$$v/c = 0.013468725x^{\frac{1}{2}} = 0.013468725y_0^{\frac{1}{2}}\left(\phi^2 - \frac{1}{y_0^2}\right. \tag{20}$$

Figures 1 and 2 illustrate the characteristic features of v/c for five selected values of the physical parameter $y_0^{-2} = 0.01$, 0.02, 0.05, 0.10 and 0.20. We note some common properties : (i) values of v_s are the highest at the centre, decrease monotonically outwards from the centre and tend to zero at the boundary of the configuration (ii) sound velocity decreases with the increases of y_0^2. In the case $x \to \infty$, (v/c, η) curves show more steeping trend for y_0^{-2} = 0.01 than those for y_0^{-2}=0, 02, 0.05, 0.10 and 0.20 (v/c attains the lowest value for y_0^{-2} = 0.20 and shows very small variations with η).

Tables 1 and 2 present a comparative study of the values of some physical parameters, such as, the boundary values η_1, mass function $\eta_1^2\phi'(\eta_1)$, ratio of central density to mean density $\rho_0/\bar{\rho}$ and the ratio of total mass to the limiting mass M/M_3, the ratio of the geometrical size R to the length l_1, R/l_1, respectively, as obtained by our present approximate analytical technique and those of previous numerical methods. The domain of the exactness of our results is measured by the relative error

$$R.E. = \frac{V_{an} - V_{nu}}{V_{nu}}$$

where V_{an} and V_{nu} denote values obtained by the "*analytical*" and "*numerical*" methods, respectively.

From Tables 1 and 2, it is clear that our approximate analytical results are very close to those of previously calculated values as indicated by the relative error R.E. formula.

6. Conclusions

(i) An analytic approach has been made to discuss the special relativistic structural features of white dwarfs governed by the differential eq. (4) under the boundary conditions (5).

(ii) Approximate analytical solution to the equilibrium equation (4) is obtained in simple form [eq. (8)] suitable for computer programming.

(iii) The main advantages of the present approach are : (i) Exhaustive computer programmes are not needed to compute the numerical values of the physical parameters; (ii) Most importantly, the boundary values η_1 of the configurations can be determined with least effort in minimum time by simply solving the biquadratic equation (10) without any necessity of making long table of ϕ and η as used in previous numerical methods; and (iii) The present method is most economical from the computational of view.

(iv) Structural parameters of the white dwarfs, such as, the ratio of mean density to central density [eq.(11)], the ratio of total radius to the length [eq. (14)] and the ratio of total mass to the limiting mass [eq. (13)] have been obtained in convenint forms. Tables 1 and 2 list the numerical values of $-\eta_1$, $\eta_1^2\phi'(\eta_1)$, $\rho_0/\bar{\rho}$ and M/M_3, R/l_1 respectively, for a new assigned values of y_0^2.

It has been found that our approximate analytical results are in good agreement with those of Chandrasekhar, computed by the numerical method. The domain of the accuracy of our approximate calculations is indicated by the relative error in Tables 1 and 2 [marked with double asterisk(**)].

(v) Figures 1-3 illustrate the characteristic features of the ratio or velocity of sound to that of light v/c (< 1, agreeing well with the physical principle of causality for both small ($x \to 0$) and large ($x \to \infty$) electronic concentrations inside the configurations and total energy E for y_0^2 = 0.01, 0.02, 0.05, 0.10 and 0.20.

(vi) Stability considerations could be another aspect of the problem which, we hope, to be able to return in future work.

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